Set Partitions
in Discrete Mathematics

1. Definition of a Set Partition

A partition of a set is a way of dividing (or "splitting") the set into smaller, **non-empty groups (called subsets)** such that:

1. Every element of the original set is in exactly one of these subsets.
2. The subsets are disjoint (they don't overlap; no element is shared between them).
3. The union of all these subsets gives back the entire original set.

In formal terms: Let S be a non-empty set. A partition P of S is a collection of non-empty subsets $A_1, A_2, \ldots, A_k$ (for some positive integer k) where:

- $A_i \cap A_j = \emptyset$ for all $i \neq j$ (disjoint).
- $A_1 \cup A_2 \cup \cdots \cup A_k = S$ (covers the whole set).

Note: The order of the subsets doesn't matter, and the subsets themselves don't have to be ordered.

2. Purpose of Set Partitions

Why do we care about partitions? They're not just for fun—they're super useful in math and real-world applications! Here's why:

Organizing Data: Partitions help group similar items. For example, in computer science, you might partition data for efficient storage or searching (like dividing files into folders).

Equivalence Relations: In discrete math, partitions are closely tied to equivalence relations (like "is congruent to" in modular arithmetic). The groups in a partition are like "equivalence classes"—elements that are "related" in some way.

Counting Problems (Combinatorics): We use partitions to solve problems like "How many ways can I divide n objects into k groups?" This leads to cool things like Bell numbers (total partitions of a set) or Stirling numbers (partitions into exactly k subsets).

Real-World Applications:

- In statistics: Partitioning data into clusters for analysis.
- In graph theory: Partitioning vertices into independent sets.
- In scheduling: Dividing tasks into non-overlapping groups.

Partitions help us think about structure and classification without overlaps or gaps, which is key in proofs, algorithms, and modeling.

3. Examples of Set Partitions

Let's make this concrete with a small set. Suppose $S = \{1, 2, 3\}$. What are all possible partitions of S?

- **Partition into 1 subset (the whole set):** $\{\{1, 2, 3\}\}$
  - This is the "trivial" partition. Everything is together.
- **Partitions into 2 subsets:**
  - $\{\{1, 2\}, \{3\}\}$
  - $\{\{1, 3\}, \{2\}\}$
  - $\{\{1\}, \{2, 3\}\}$
  - (Note: These are all the ways to split into two non-empty groups.)
- **Partition into 3 subsets (singletons):** $\{\{1\}, \{2\}, \{3\}\}$
  - Everything is separate.

Total partitions: 5. (This is the Bell number for n=3.)

Another Example: Let $T = \{a, b, c, d\}$ (a set with 4 elements).

- One partition: $\{\{a, b\}, \{c\}, \{d\}\}$ — Groups of size 2, 1, 1.
- Another: $\{\{a, c, d\}, \{b\}\}$ — One big group and one singleton.
- Not a partition: $\{\{a, b\}, \{b, c\}\}$ — Why? Overlap (b is in both), and d is missing.

Which of the following is a partition of set U={1,2,3,......9}.
(i){{1,2},{3,4,5},{5,6},{7,8},{9}}
(ii) {{1,2,3},{4,5},{6,7}}
(iii) {{1,2,3,4}.{5,6},{7,8,9}}

Consider the set S={1,2,3,..........,14,15} and its subsets A1={3,6,9,12,15}, A2={2,5,8,11,14},A3={1,4,7,10,13}.Is the set {A1,A2,A3} a partition of S.

Question 1 (5 marks)

Define what a partition of a set S is. Your definition must clearly state the three key properties that the collection of subsets must satisfy.

Question 2 (6 marks)

Consider the set S = {1, 2, 3, 4}.

For each of the following collections, state whether it is a valid partition of S. If it is not, explain which property (or properties) of a partition is violated.

(a) {{1, 2}, {3, 4}}
(b) {{1}, {2, 3}, {3, 4}}
(c) {{1, 2, 3, 4}}
(d) {{1, 3}, {2}, {4}, ∅}

Question 3 (8 marks)
List all possible partitions of the set A = {a, b, c}.
(You may write them using any clear notation, e.g., {{a,b},{c}}.)
How many partitions are there in total?

Question 4 (6 marks)
True or False? Justify your answer briefly in each case.
(a) The collection {{1, 2}, {2, 3}} is a partition of {1, 2, 3}.
(b) Every set has exactly two partitions: one consisting of all singletons and one consisting of the whole set itself.
(c) The empty collection ∅ is a partition of the empty set.

Equivalence Classes

An **equivalence class** is the group of all elements in a set that are "equivalent" to a particular element under an **equivalence relation** (a relation that is reflexive, symmetric, & transitive).

Let's take the set $S = \{1, 2, 3, 4, 5\}$.

We define the relation ~ explicitly as follows:

$$= \{(1,1), (1,3), (1,5), (2,2), (2,4), (3,1), (3,3), (3,5), (4,2), (4,4), (5,1), (5,3), (5,5)\}$$

In words: $a \sim b$ if and only if a and b have the same parity (both even or both odd), i.e., congruent modulo 2.

Verification that ~ is an Equivalence Relation

- **Reflexive:** Every element is related to itself — (1,1), (2,2), (3,3), (4,4), (5,5) are all in ~.
- **Symmetric:** For every pair like (1,3), the reverse (3,1) is also in ~ (similarly for all others).
- **Transitive:** If $a \sim b$ and $b \sim c$, then $a \sim c$ (e.g., 1 ~ 3 and 3 ~ 5 ⇒ 1 ~ 5).

Since ~ satisfies all three properties, it is an **equivalence relation.**

Equivalence Classes

The equivalence classes are:

- $[1] = \{1, 3, 5\}$ (all odd numbers)
- $[2] = \{2, 4\}$ (all even numbers)

(Note: $[3] = [1]$, $[4] = [2]$, $[5] = [1]$. We list only the distinct classes.)

These classes are disjoint, non-empty, and their union is S, so they form a **partition** of S.